October Math Gems

Problem of the week 21

§1 problems

Problem 1.1. Prove that

$$\csc 6^\circ + \ csc78^\circ - \csc 42^\circ - \csc 66^\circ = 8$$

answer. $T = \csc 6^\circ + \csc 78^\circ - \csc 42^\circ - \csc 66^\circ$. $T = \frac{1}{\sin 6^\circ} + \frac{1}{\sin 78^\circ} - \frac{1}{\sin 42^\circ} - \frac{1}{\sin 66^\circ}$.
$T = \frac{\sin 6^\circ + \sin 78^\circ}{\sin 6^\circ \sin 78^\circ} - \frac{2\cos 42^\circ}{2\sin 42^\circ \cos 42^\circ} - \frac{1}{\cos 24^\circ}$.
$T = \frac{2\sin 42^\circ \cos 36^\circ}{\sin 6^\circ \sin 78^\circ} - \frac{2\cos 42^\circ}{\sin 84^\circ} - \frac{2\sin 24^\circ}{2\sin 24^\circ \cos 24^\circ}$.
$T = \frac{2\sin 42^\circ \cos 36^\circ}{\sin 6^\circ \cos 12^\circ} - \frac{2\cos 42^\circ}{\cos 6^\circ} - \frac{2\sin 24^\circ}{\sin 48^\circ}$.
$T = \frac{2\sin 42^\circ(4\cos^3 12^\circ - 3\cos 12^\circ)}{\sin 6^\circ \cos 12^\circ} - \frac{4\sin 6^\circ \cos 42^\circ}{2\sin 6^\circ \cos 6^\circ} - \frac{4\sin 24^\circ \cos 48^\circ}{2\sin 48^\circ \cos 48^\circ}$.
$T = \frac{2\sin 42^\circ(4\cos^2 12^\circ - 3)}{\sin 6^\circ} - \frac{2(\sin 48^\circ - \sin 36^\circ)}{\sin 12^\circ} - \frac{4\sin 24^\circ \cos 48^\circ}{\sin 96^\circ}$.
$T = \frac{2\sin 42^\circ(2\cos 24^\circ + 2 - 3)}{\sin 6^\circ} - \frac{2(4\sin 12^\circ \cos 12^\circ \cos 24^\circ - (3\sin 12^\circ - 4\sin^3 12^\circ)}{\sin 12^\circ} - \frac{16\sin 6^\circ \cos 6^\circ \cos 12^\circ \cos 48^\circ}{\cos 6^\circ}$.
$T = \frac{2(\sin 66^\circ + \sin 18^\circ - \sin 42^\circ)}{\sin 6^\circ} - 2(4\cos 12^\circ \cos 24^\circ - (3 - 4\sin^2 12^\circ) - 16\sin 6^\circ \cos 12^\circ \cos 48^\circ$.
$T = \frac{2(2\sin 12^\circ \cos 54^\circ + \sin 18^\circ)}{\sin 6^\circ} - 2(4\cos 12^\circ \cos 24^\circ - (3 - 4\sin^2 12^\circ) - 16\sin 6^\circ \cos 12^\circ \cos 48^\circ$.
$T = \frac{2(4\sin 6^\circ \cos 6^\circ \cos 54^\circ + 3\sin 6^\circ - 4\sin^3 6^\circ)}{\sin 6^\circ} - 2(4\cos 12^\circ \cos 24^\circ - (3 - 4\sin^2 12^\circ) - 16\sin 6^\circ \cos 12^\circ \cos 48^\circ$.
$T = 2(4\cos 6^\circ \cos 54^\circ + 3 - 4\sin^2 6^\circ) - 2(4\cos 12^\circ \cos 24^\circ - (3 - 4\sin^2 12^\circ) - 16\sin 6^\circ \cos 12^\circ \cos 48^\circ$.
$T = 2[4\cos 6^\circ \cos 54^\circ + 3 - 2(1 - \cos 12^\circ) - 4\cos 12^\circ \cos 24^\circ + 3 - 2(1 - \cos 24^\circ) - 8\sin 6^\circ \cos 12^\circ \cos 48^\circ]$.
$T = 2[2\cos 60^\circ + 2\cos 48^\circ + 1 + 2\cos 12^\circ - 2\cos 36^\circ - 2\cos 12^\circ + 1 + 2\cos 24^\circ - 4(\sin 18^\circ - \sin 6^\circ)\cos 48^\circ]$.
$T = 4[\cos 60^\circ + \cos 48^\circ + 1 - \cos 36^\circ + \cos 24^\circ - 2(\sin 18^\circ - \sin 6^\circ)\cos 48^\circ]$.
$T = 4[\cos 60^\circ + \cos 48^\circ + 1 - \cos 36^\circ + \cos 24^\circ - \sin 66^\circ + \sin 30^\circ + \sin 54^\circ - \sin 42^\circ]$.
$T = 4[2 - \cos 36^\circ + \cos 24^\circ - \sin 66^\circ + \sin 54^\circ]$.
$T = 4[2 - 2\sin 30^\circ \sin 6^\circ + 2\sin 6^\circ \cos 60^\circ] \ = \ 8$. $\square$

Problem 1.2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

answer. let $p(x,y) = f(\frac{y}{f(x+1)}) + f(\frac{x+1}{xf(y)}) = f(y)$ if we have y such that $yf(y) > 1$,$x = \frac{1}{yf(y)-1} \Rightarrow f(\frac{y}{f(x+1)}) = 0$ which is not true.
so for all x we have $f(x) \le \frac{1}{x}$. $f(y) = \frac{y}{f(x+1)}) + f(\frac{x+1}{xf(y)}) \le \frac{x}{x+1}f(y) + f(\frac{y}{f(x+1)}) \Rightarrow \frac{f(y)}{x+1} \le f(\frac{y}{f(x+1)}) \le f(x+1).\frac{1}{y} \Rightarrow yf(y) \le (x+1)f(x+1) \Rightarrow$
for all $1 < x$ $f(x) = \frac{a}{x} \to yf(y) \le a \le 1$ $p(x, y+1) \Rightarrow a = 1$
$p(x,1) \Rightarrow f(1) = 1$ $p(x, \frac{1}{x+1}), (\frac{x+1}{xf(\frac{1}{x+1})} > 1) \Rightarrow f(\frac{1}{x+1}) = x + 1 \Rightarrow f(x) = \frac{1}{x}$ $\square$

Problem 1.3. Find $f : \mathbb{R} \to \mathbb{R}$ which satisfied:

$$x^2 f(y) + y^2 f(x) = xyf(x + y)$$

answer. Let $P(x, y)$ be the assertion $x^2 f(y) + y^2 f(x) = xyf(x + y)$ Let $c = f(1)$ $P(1, 1) \implies f(2) = 2c$
$P(x, 1) \implies cx^2 + f(x) = xf(x+1)$ $P(x+1, 1) \implies c(x+1)^2 + f(x+1) = (x+1)f(x+2)$
$P(x, 2) \implies 2cx^2 + 4f(x) = 2xf(x + 2)$
Multiplying the second by x and the third by $\frac{x+1}{2}$, this becomes (a) : $cx^2+f(x) = xf(x+1)$ (b) : $cx(x+1)^2+xf(x+1) = x(x+1)f(x+2)$ (c) : $cx^2(x+1)+2(x+1)f(x) = x(x+1)f(x+2)$
(a)+(b)-(c) : $(2x + 1)f(x) = cx(2x + 1)$ and so $f(x) = cx$ $\forall x \neq -\frac{1}{2}$
Then $P(-\frac{1}{2}, -\frac{1}{2}) \implies f(-\frac{1}{2}) = -\frac{c}{2}$ And so $\boxed{f(x) = cx \quad \forall x}$, which indeed fits, whatever is $c \in \mathbb{R}$ □

Problem 1.4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\left(x\left(f(x) - \frac{f(y) + f(z)}{2}\right) + y\left(f(y) - \frac{f(z) + f(x)}{2}\right) + z\left(f(z) - \frac{f(x) + f(y)}{2}\right)\right) f(x + y + z) = f(x^3) + f(y^3) + f(z^3) - 3f(xyz)$$

for all $x, y, z \in \mathbb{R}$.

answer. Let $P(x, y, z)$ be the assertion $\frac{1}{2}(f(x)(2x - y - z) + f(y)(2y - x - z) + f(z)(2z - x - y))f(x + y + z) = f(x^3) + f(y^3) + f(z^3) - 3f(xyz)$
Let $a = f(0)$, $c = f(1)$ and $d = f(-1)$ $P(1, 0, 0) \implies (c - 1)(c - a) = 0$ $P(-1, 0, 0) \implies (d+1)(d-a) = 0$ $P(1, 1, -1) \implies (c-1)(c-d) = 0$ $P(1, -1, -1) \implies (d+1)(c-d) = 0$
And so either $c = 1$ and $d = -1$, either $c = d = a$
1) If $c = d = a$ $P(x, 1, -1) \implies x(f(x) - a)f(x) = f(x^3) + 2a - 3f(-x)$ $P(x, 0, 0) \implies f(x^3) = xf(x)^2 - axf(x) + a$ Adding, we get $f(-x) = a$ and so $\boxed{\text{S1} : f(x) = a \quad \forall x}$, which indeed fits, whatever is $a \in \mathbb{R}$
2) If $c = 1$ and $d = -1$ $P(0, 1, -1) \implies a = 0$ $P(x, 0, 0) \implies f(x^3) = xf(x)^2$ $P(x, 1, -1) \implies f(-x) = -f(x)$ $P(x, y, -y) \implies f(xy^2) = yf(x)f(y)$
Setting there $x = 1$, we get $f(y^2) = yf(y)$ and so $f(xy^2) = f(x)f(y^2)$ So $f(xy) = f(x)f(y)$ $\forall y \geq 0$ and so, since odd, $\forall x, y$ So $f(x^3) = f(x)^3$ and so $f(x)^3 = xf(x)^2$ So $\forall x$, either $f(x) = 0$, either $f(x) = x$
If $f(u) = 0$ for some $u \neq 0$, $1 = f(1) = f(u\frac{1}{u}) = f(u)f(\frac{1}{u}) = 0$, contradiction And so $\boxed{\text{S2} : f(x) = x \quad \forall x}$, which indeed fits. □

Problem 1.5. In a triangle ABC, $\frac{cosA}{1+sinA} = \frac{sin2B}{1+cos2B}$. Find the minimum value of $\frac{a^2+b^2}{c^2}$.

answer. $\frac{sin2B}{1+cos2B} = \tan B$ $\frac{cosA}{1+sinA} = \frac{\sin(\frac{\pi}{2}-A)}{1+\cos(\frac{\pi}{2}-A)} = \tan(\frac{\pi}{4} - \frac{A}{2})$
$\tan B = \tan(\frac{\pi}{4} - \frac{A}{2}) \to B = \frac{\pi}{4} - \frac{A}{2}, A < \frac{\pi}{2}$ or $B - \pi = \frac{\pi}{4} - \frac{A}{2}, A > \frac{\pi}{2} \to B = \frac{5\pi}{4} - \frac{A}{2} \to A + B = \frac{5\pi+2A}{4} > \frac{3\pi}{2} \to$ contradiction. So $B = \frac{\pi}{4} - \frac{A}{2}, A = \frac{\pi}{2} - 2B, C = \pi - A - B = \frac{3\pi}{4} - \frac{A}{2} = B + \frac{\pi}{2}$ where $B < \frac{\pi}{4}$

$$\frac{a^2 + b^2}{c^2} = \frac{\sin^2 A + \sin^2 B}{\sin^2 C} = \frac{\cos^2 2B + \sin^2 B}{\cos^2 B} = \frac{(2\cos^2 B - 1)^2 + 1}{\cos^2 B} - 1 = 4\cos^2 B + \frac{2}{\cos^2 B} + 3 \geq 4\sqrt{2} + 3$$

with equality for

$$\cos B = \frac{1}{\sqrt[4]{2}}$$

□

Problem 1.6. Find all polynomials

$$(x-1)p(2x) = 4(x+1)p(x) + 5$$

answer. Setting $x = 0$, we get $P(0) = -1$.
Let then ax^n with $a \neq 0$ the greatest degree summand of $P(x)$
Greatest degree summand of LHS is $a2^n x^{n+1}$
Greatest degree summand of RHS is $4ax^{n+1}$
So $n = 2$ and $P(x) = ax^2 + bx - 1$

Plugging this in original equation, we get $b = \frac{1}{2}$ and then $a = -\frac{1}{8}$ And so $\boxed{P(x) = -\dfrac{x^2}{8} + \dfrac{x}{2} - 1}$ □

Problem 1.7.

$$\log_{2x}^2 \left(4x^3\right) - 2 = \log_{2x} 4x$$

answer. First, note that must hold $x > 0$ and $x \neq \frac{1}{2}$ in order to logarithms be defined. $\log_{2x}^2 \left(4x^3\right) - 2 = \log_{2x}(4x)$ $\left[\log_{2x}\left(8x^3\right) - \log_{2x} 2\right]^2 - 2 = \log_{2x}(2x) + \log_{2x} 2$ $(3 - \log_{2x} 2)^2 - 2 = 1 + \log_{2x} 2$ $(\log_{2x} 2)^2 - 7\log_{2x} 2 + 6 = 0$ $\log_{2x} 2 = 1$ or $\log_{2x} 2 = 6$ $x = 1$ or $x = \dfrac{\sqrt[6]{2}}{2}$ □

Problem 1.8. Determine all the real numbers x for which

$$2\log_2(x-1) = 1 - \log_2(x+2)$$

answer. $2\log_2(x-1) = 1 - \log_2(x+2)$ $\log_2(x-1)^2 = \log_2 2 - \log_2(x+2)$ $\log_2(x-1)^2 = \log_2 \frac{2}{x+2}$ $(x-1)^2 = \frac{2}{x+2}$ $(x-1)^2(x+2) - 2 = 0$ $x^3 - 3x = 0$, solution only $x = \sqrt{3}$. □

Problem 1.9. Find the number of solution of $\log_4(x-1) = \log_2(x-3)$.

answer. Let this common value be y. Then, $x - 1 = 4^y = (2^y)^2 = (x-3)^2 \implies (x-2)(x-5) = 0$, we can check that $x = 2$ is extraneous and so the only solution is $x = 5$ □

Problem 1.10. Given that :

$$log_4(log_{\sqrt{2}}x) + log_{\sqrt{2}}(log_4 x) = \frac{7}{2}$$

Find the value of:

$$log_2 2x + log_8 \frac{x}{2}$$

answer. Set $x = 2^{2^a}$ $log_4(log_{\sqrt{2}}x) + log_{\sqrt{2}}(log_4 x) = log_4(2^{a+1}) + log_{\sqrt{2}}(2^{a-1}) = \frac{a+1}{2} + 2(a-1) = \frac{7}{2}$ $a = 2$ $x = 16$ $log_2 2x + log_8 \frac{x}{2} = 6$ □

Problem 1.11. Solve over $\mathbb{R}$ the equation $4^{(sinx)^2} + 3^{(tanx)^2} = 4^{(cosx)^2} + 3^{(cotanx)^2}$.

answer. Setting $t = \cos^2 x \in (0,1)$, this is $4^{1-t} + 3^{\frac{1-t}{t}} = 4^t + 3^{\frac{t}{1-t}}$ LHS is decreasing while RHS is increasing and so at most one real root over $(0,1)$ $t = \frac{1}{2}$ is a trivial root and so is the only one. And so $\cos^2 x = \frac{1}{2}$ which is $\boxed{x = \dfrac{\pi}{4} + k\dfrac{\pi}{2}}$ □

Problem 1.12. Solve in $\mathbb{R}$ the equation $\frac{1}{2^x+x+1}+\frac{1}{3^x-4x-3}=\frac{1}{2^x-4x-2}+\frac{1}{x+3^x}$.

answer. Common denominator on both sides yields

$$\frac{3^x-3x-2+2^x}{(2^x+x+1)(3^x-4x-3)}=\frac{3^x-3x-2+2^x}{(2^x-4x-2)(x+3^x)}$$

We see that the numerators are equal, meaning that this equality holds when the numerator is equivalent to 0, or when the denominators are equal and not 0. Checking the first case, $3^x-3x-2+2^x=0$ in real numbers when $x=0,1$. The second case, we solve as

$$(2^x+x+1)(3^x-4x-3)=(2^x-4x-2)(x+3^x)$$

$$(2^x)(3^x)-(2^x)(4x)-(3)(2^x)+(x)(3^x)-4x^2-3x+3^x-4x-3=(2^x)(3^x)-(4x)(3^x)-(2)(3^x)+(x)(2^x)-4x^2-2x$$

$$(4x)(3^x)-(4x)(2^x)+(x)(3^x)-(x)(2^x)+(3)(3^x)-(3)(2^x)-5x-3=0$$

$$(3^x-2^x-1)(5x+3)=0$$

This is true when $x=-\frac{3}{5},1$. Checking our answers through substitution verifies that our solutions are indeed $\boxed{x=0,1,-\frac{3}{5}}$. □

Problem 1.13. Solve in real numbers the following equation: $(\sqrt{6}-\sqrt{5})^x+(\sqrt{3}-\sqrt{2})^x+(\sqrt{3}+\sqrt{2})^x+(\sqrt{6}+\sqrt{5})^x=32$.

answer. Let $a=\sqrt{6}-\sqrt{5}$ and $b=\sqrt{3}-\sqrt{2}$ and $f(x)=a^x+a^{-x}+b^x+b^{-x}-32$ $f(x)$ is an even function increasing over $\mathbb{R}^+$ and so decreasing over $\mathbb{R}^-$ So at most one real non negative root and its opposite. And since $f(2)=0$ we get the answer $\boxed{x\in\{-2,+2\}}$ □

Problem 1.14. Prove that $1-\frac{sin^2\alpha}{1+cot\alpha}-\frac{cos^2\alpha}{1+tan\alpha}=cos\alpha.sin\alpha$

answer. $1-\frac{\sin^2\alpha}{1+\cot\alpha}-\frac{\cos^2\alpha}{1+\tan\alpha}=1-\frac{\sin^3\alpha}{\cos\alpha+\sin\alpha}-\frac{\cos^3\alpha}{\sin\alpha+\cos\alpha}=1-\sin^2\alpha-\cos^2\alpha+\sin\alpha\cos\alpha=\sin\alpha\cos\alpha$ □

Problem 1.15. Prove that $\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}$

answer. $\cos x-\cos 2x+\cos 3x=\frac{1}{2}$ $\cos x+\cos 3x+\cos 5x=\frac{1}{2}$ $\cos x+\cos 3x+\cos 5x+\ldots=\frac{\sin 2nx}{2\sin x}$ $n=3$ $\cos x+\cos 3x+\cos 5x=\frac{\sin 6x}{2\sin x}$ $\frac{\sin 6x}{2\sin x}=\frac{1}{2}\Leftrightarrow\sin 6x=\sin x$ $x=\frac{2k\pi}{5}(1)$ $x=\frac{(2k+1)\pi}{7}(2)$ (2) $k=0\rightarrow x=\frac{\pi}{7}$ $\therefore$ $\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}$ □

Problem 1.16. Determine all real numbers $x>0$ for which

$$\log_4 x-\log_x 16=\frac{7}{6}-\log_x 8$$

answer. We have given the equation (1) $\log_4 x-\log_x 16=\frac{7}{6}-\log_x 8$, Clearly $x>0$ by equation (1), which implies there is a real number y s.t. (2) $x=2^y$. The fact that $x=(2^n)^{\frac{y}{n}}$ when $n\in\mathbb{N}$ yields $\log_{2^n}x=\frac{y}{n}$, which combined with equation (1) give us $\frac{y}{2}-\log_x 2^4=\frac{7}{6}-\frac{y}{3}$, i.e. (3) $5y-7=24\log_x 2$. Now there is a real number z s.t. $x^z=2$, which according to formula (2) means $2^{yz}=2$, implying $yz=1$. The fact that $x^z=2$ implies $\log_x 2=z=\frac{1}{y}$, which inserted in equation (3) result in $5y-7=\frac{24}{y}$, yielding $5y^2-7y+24=0$, i.e. $(5y+8)(y-3)=0$. Consequently $y=-\frac{8}{5}$ or $y=3$, which according to formula (2) means there are exactly two solutions of equation (1), namely $x=2^{-\frac{8}{5}}=\frac{1}{\sqrt[5]{256}}$ and $x=2^3=8$. □

Problem 1.17. Solve the equation $x^{\log_{10} x} = 3y/2$

answer.

$$\log_{10} x^{\log_{10} x} = log_{10}\frac{3y}{2} \Longrightarrow \log_{10} x = \sqrt{log_{10}\frac{3y}{2}} \Longrightarrow x = 10^{\sqrt{log_{10}\frac{3y}{2}}}$$

□

Problem 1.18. Find number of solutions to the equation $\log_{x+1}(x - \frac{1}{2}) = \log_{x-\frac{1}{2}}(x + 1)$

answer. $\log_{x+1}(x-\frac{1}{2}) = \log_{x-\frac{1}{2}}(x+1) \Leftrightarrow \log_{x+1}(x-\frac{1}{2}) = \frac{1}{\log_{x+1}(x-\frac{1}{2})} \Leftrightarrow \left(\log_{x+1}(x - \frac{1}{2})\right)^2 = 1$, so $\log_{x+1}(x - \frac{1}{2}) = \pm 1 \Rightarrow 2x^2 + x - 3 = 0 \Rightarrow x \in \left\{1, \frac{3}{2}\right\}$ □

Problem 1.19. A¡=b¡=c $a^2 + b^2 = c^2$ A,b,c are sides of a traingle $loga^2 + logb^2 = logc^2$ Find Max A

answer. We have $a^2 + b^2 = c^2, a^2b^2 = c^2 \Rightarrow a^2 + b^2 = a^2b^2 \Leftrightarrow a^2 = 1 + \frac{1}{b^2-1} \leq 1 + \frac{1}{a^2-1} \Rightarrow$ $\max A = B = 45°$ □

Problem 1.20. Solve for x

$$log_2 x = log_{5-x} 3$$

answer. By change of bases: $\frac{\log x}{\log 2} = \frac{\log 3}{\log 5-x} \Longrightarrow \log x \cdot \log(5 - x) = \log 2 \cdot \log 3$ $\Longrightarrow \log(5x - x^2) = \log 6$ Therefore: $5x - x^2 = 6 \Longrightarrow x^2 - 5x + 6 = 0$ $(x - 3)(x - 2) = 0$ $\Longrightarrow \boxed{x = 2, 3}$ □